2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Formal stabilization of a coupled ODE-PDE switched system

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Abstract— **Partial Differential Equations (PDEs) are a ubiquitous model that describes a wide range of dynamical systems. While stabilization of systems involving PDEs is an important problem, little work has been done on formal verification and synthesis for such systems. In this paper, we explain how a coupled ODE-PDE control problem can be formally stabilized using a tiling based control synthesis algorithm associated to set based reachability, which is usually used on finite dimensional problems. To formally prove the stabilization of the PDE, the original infinite dimensional problem is transformed into a finite dimensional one using, among other tools, model order reduction. The strength of our approach relies on the fact that we never explicitly discretize the PDE state using *e.g.* a finite element approximation, and consequently, we provide stability guarantees directly on the infinite dimensional state.**

I. Introduction

Stabilization of infinite-dimensional control systems, *i.e.* governed by partial differential equations (PDEs), is a widely researched topic [21], with numerous mathematical results ensuring *e.g.* exponential convergence of the infinite dimensional state towards zero. Extensions of such methods to the case of switched controls exist, but are scarcer [15], [20], [14]. When it comes to providing formal guarantees for such systems, results are extremely rare. A first reason lies in the computational and theoretical difficulties of the task. A second reason lies in the amount of multi-disciplinary work it requires, spanning multiple domains such as computer science, mathematics and computational mechanics.

It is well known that switched ordinary differential equations can be controlled with correct–by–construction methods, symbolic and abstraction methods being the most common. They can rely on state abstraction [19], [4], [22], hybridization [2], or tiling [10]. Spatiotemporal discretization methods for infinite dimensional systems such as finite difference or Finite Element Methods (FEM) are widely used for simulation of PDEs. These methods allow to transform PDEs into finite (high) dimensional systems, that can sometimes be reduced into low dimensional systems with dimension reduction techniques, also called model order reduction (MOR) methods [5]. These methods are usually provided with error estimates, and sometimes error bounds for the state discretized (high dimensional) system [13]. While the FEM is associated with known error estimates [3], and theoretical error bounds (with unknown constants) have long been available [8], proper computable error bounds are still hard to obtain [1]. Given these difficulties, formal reachability analysis for a parabolic system has been attempted in [23] using a Galerkin FEM, but the method was not fully conservative since the authors could not formally bound the approximation error. Another formal verification approach relying on the FEM has been proposed in [18] using Signal Temporal Logic, but the approximation error is hypothesized as bounded.

Contributions: In this paper, we propose an approach giving formal stability guarantees for a coupled ODE-PDE switched system directly on the infinite dimensional state. Since we do not use a space discretization scheme like the FEM, the formal guarantee is given directly on the infinite dimensional state. The system is, in a few words, a switched ordinary differential equation, whose state corresponds to the boundary conditions of the heat equation. The problem is instantiated on a switched two dimensional ODE coupled with the heat equation. Our approach relies on a tiling–based control synthesis method. In order to apply this method to the coupled ODE-PDE system, we leverage several mathematical transformations (principle of superposition, model reduction, etc.) to approximate the system by a numerically achievable finite dimensional system. The main idea is to decompose the infinite dimensional state into the sum of multiple terms exhibiting different behaviors, some reducible to finite dimension and formally controllable while others exhibit limiting factors (or control guidelines). Here, we focus on the methods used to formally handle this control problem, the mathematical proofs are left in the appendices. The model reduction technique used here is *ad-hoc*, although, advances in computational mechanics provide more and more error estimates and guaranteed bounds that are usable in formal methods. Thus, our work provides a correct–by–construction method for handling a given type of ODE-PDE system that can be extended to many others upon using an adapted model reduction method.

The paper is organised as follows. In Section II, we formally define the coupled ODE-PDE control problem. In Section III, we introduce the control synthesis algorithm that is used for ODE control problems. In Section IV, we explain how the control synthesis algorithm can be used for guaranteeing the stabilization of the coupled ODE-PDE problem. For the sake of readability, the proofs of the main mathematical results are left in the appendices.

Notations: Let $\Omega \subset \mathbb{R}$. Let $L^2 = L^2(\Omega)$ be the space of measurable functions $f : \Omega \to \mathbb{R}$ whose square is Lebesgue integrable. Let $H^1 = H^1(\Omega)$ be the subspace of L^2 of functions whose weak derivatives are also in L^2. Let $H_0^1 = H_0^1(\Omega)$ be the closure in H^1 of the space of infinitely differentiable

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compactly supported functions. Let $L^\infty = L^\infty(\Omega)$ be the space of measurable functions that are bounded almost everywhere (essentially bounded). Finally, let $L^1_{\text{loc}}([0,\infty[;H^1)$ be the Bochner space of functions $f(x,t)$ such that $f(\cdot,t) \in H^1$ for any $t \geq 0$ and $t \mapsto \|f(\cdot,t)\|_{H^1}$ is in $L^1_{\text{loc}}([0,\infty[)$, where $L^1_{\text{loc}}([0,\infty[)$ denotes the space of locally integrable functions, that is, the functions which are Lebesgue integrable on every compact subsets of $[0,\infty[$. For a vector $\boldsymbol{z} \in \mathbb{R}^n$ and $r > 0$, we denote by $B(\boldsymbol{z},r)$ the ball of center $\boldsymbol{z}$ and radius r with Euclidian norm, and by $B_\infty(\boldsymbol{z},r)$ the ball of center $\boldsymbol{z}$ and radius r with ℓ^∞ norm. For a function $f \in L^2$ and $r > 0$, we denote by $B_{L^2}(f,r)$ the ball of center f and radius r with L^2 norm.

II. Problem definition

Let $L > 0$ and set $\Omega =]0,L[$ the spatial domain of the PDE. Let $\kappa \in L^\infty(\Omega)$, and suppose there exist two constants κ_m and κ_M, $0 < \kappa_m \leq \kappa_M$ such that

$$\kappa_m \leq \kappa(x) \leq \kappa_M \text{ for } x \text{ in } \Omega. \tag{1}$$

Let $U = \{1,\ldots,M\}$ be the set of switched modes, $A_1, \ldots, A_M \in \mathbb{R}^{2\times 2}$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_M \in \mathbb{R}^2$, the space of admissible switched control sequences is

$$\Sigma^\tau = \left\{\sigma : [0,+\infty[\to U, \sigma|_{[q\tau,(q+1)\tau[}(t) \in U,\ \forall q \in \mathbb{N}\right\}. \tag{2}$$

We consider the one-dimensional boundary switched control heat problem with arbitrary source term $f \in H^1$ that is to find a piecewise constant sequence $\sigma \in \Sigma^\tau$, such that the vector–valued state $\boldsymbol{\xi} \in C^0([0,\infty[)^2$ (continuous functions) and the functions $u \in L^1_{\text{loc}}([0,\infty[;H^1)$ that are solutions to the problem

$$\begin{cases} \text{for all } (x,t) \in \Omega \times [0,+\infty[, \\ \dot{\boldsymbol{\xi}}(t) = A_\sigma \boldsymbol{\xi}(t) + \boldsymbol{b}_\sigma(t) \quad \text{with} \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}^0, \\ \dfrac{\partial u}{\partial t}(x,t) - \nabla \cdot (\kappa \nabla u)(x,t) = f(x,t), \\ \text{with} \quad u(0,t) = \xi_1(t),\, u(L,t) = \xi_2(t) \,\text{and}\, u(x,0) = u^0(x) \end{cases} \tag{3}$$

verify, for any initial conditions $\boldsymbol{\xi}^0 \in \mathbb{R}^2$ and $u^0 \in H^1$ and objective $u^\infty \in H^1$, the stability constraints

$$\begin{cases} \boldsymbol{\xi}(t) \in S_\xi \quad \text{for all } t > 0, \\ \|u(.,t) - u^\infty\|_{L^2} \leq \rho \quad \text{for all } t > 0. \end{cases} \tag{4}$$

Thus the expected stability set for the global state $(\boldsymbol{\xi}(t), u(.,t))$ is the product set $S_\xi \times B_{L^2}(u^\infty,\rho) \subset \mathbb{R}^2 \times L^2$. The sequence σ will depend on the state of the system itself in order to enforce stability in the product recurrence set. The control problem is formalized as follows:

Problem 1 (ODE-PDE stability control problem): Let us consider the system (3). Given a set S_ξ, a tolerance ρ and an objective state u^∞, find a control sequence $\sigma(\boldsymbol{\xi},u) \in \Sigma^\tau$ such that, for all $t > 0$ and for all $(\boldsymbol{\xi}^0, u^0) \in S_\xi \times B_{L^2}(u^\infty,\rho)$, we have $(\boldsymbol{\xi}(t), u(.,t)) \in S_\xi \times B_{L^2}(u^\infty,\rho)$.

III. Correct-by-construction control synthesis using tiling and set-based reachability

In this section, we consider a finite dimensional control problem on a switched ODE with a generic state vector denoted by $\boldsymbol{z}$ and we detail how to control it. The state vector $\boldsymbol{z}$ will be explicitly given in section IV when we apply this method to the main problem 1.

A. Tiling-based control synthesis

Let us consider in this section a finite dimensional switched system such that

$$\dot{\boldsymbol{z}}(t) = f_{\sigma(t)}(\boldsymbol{z}(t)) \tag{5}$$

is defined for all $t \geq 0$, where $\boldsymbol{z}(t) \in \mathbb{R}^n$ is the state of the system, $\sigma \in \Sigma^\tau$ is the switching rule. The finite set $U = \{1,\ldots,M\}$ is the set of switching modes of the system.

We call "*pattern*" a finite sequence of modes $\pi = (\sigma_1,\sigma_2,\ldots,\sigma_k) \in U^k$. With such a control pattern, we will denote by $\boldsymbol{z}(t;t_0,\boldsymbol{z}_0,\pi)$ the solution at time $t \geq t_0$ of the system

$$\begin{aligned} \dot{\boldsymbol{z}}(t) &= f_{\sigma(t)}(\boldsymbol{z}(t)), \\ \boldsymbol{z}(t_0) &= \boldsymbol{z}_0, \\ \forall j \in \{1,\ldots,k\},\ \sigma(t) &= \sigma_j \in U \text{ for } t \in [t_0 + (j-1)\tau, t_0 + j\tau[. \end{aligned} \tag{6}$$

Before introducing the main controller synthesis algorithm, let us introduce some preliminary definitions.

Definition 1: Let $X \subset \mathbb{R}^n$ be a bounded set of the state space. Let $\pi = (\sigma_1,\sigma_2,\ldots,\sigma_k) \in U^k$. The *successor set* of X via π, denoted by $Post_\pi(X,t)$, is the image of X induced by application of the pattern π, *i.e.*:

$$Post_\pi(X,t) = \bigcup_{\boldsymbol{z}_0 \in X} \boldsymbol{z}(t;t_0,\boldsymbol{z}_0,\pi).$$

Definition 2: Let $X \subset \mathbb{R}^n$ be a bounded set of the state space. Let $\pi = (\sigma_1,\sigma_2,\ldots,\sigma_k) \in U^k$. We denote by $Tube_\pi(X)$ the set covering all the trajectories of system (6) for any initial conditions in X during application of pattern π, *i.e.*:

$$Tube_\pi(X) = \bigcup_{t \in [t_0,t_0+k\tau]} Post_\pi(X,t).$$

In a few words, $Post_\pi(X,t)$ is the set that encloses the state at time t reached after application of pattern π for any initial condition in $\boldsymbol{z}_0 \in X$, while $Tube_\pi(X)$ is the set that encloses all possible trajectories starting in X when applying pattern π. Note that if $t < t_0 + k\tau$ the pattern π has only been partially applied. When $t > t_0 + k\tau$ a new pattern π' must be applied. In practice, $Post_\pi(X,t)$ and $Tube_\pi(X)$ are hard to compute (especially for nonlinear systems), they are thus computed as "over-approximations" in order to guarantee rigorous results. Their computation is detailed in Section III-B.

Given a "recurrence set" $R \subset \mathbb{R}^n$ and a "safety set" $S \subset \mathbb{R}^n$ which contains R ($R \subseteq S$), the main synthesis algorithm is used to solve the following control problem: starting from any initial point $\boldsymbol{z}_0 \in R$, the controlled trajectory returns to R infinitely often while never leaving S. We suppose that sets R and S are compact. Furthermore, we suppose that S is convex. This problem is formalized as follows.

Problem 2: Given a switched system of the form (6), a recurrence set $R \subset \mathbb{R}^n$ and a safety set $S \subset \mathbb{R}^n$, find a control rule $\sigma(z) \in \Sigma^\tau$ such that, for any initial condition $z_0 \in R$, the following holds:

- Recurrence in R: there exists a monotonically strictly increasing sequence of (positive) integers $\{m_l\}_{l\in\mathbb{N}}$ such that for all $l \in \mathbb{N}$, $z(m_l\tau; t_0, z_0, \sigma) \in R$;
- Stability in S: for all $t \in \mathbb{R}^+, z(t;t_0,z_0,\sigma) \in S$.

We now describe the principle of the algorithm solving Problem 2. Given the input sets R and S, the algorithm provides, when it succeeds, a finite set of indices I, a family of sets $\{W_i\}_{i\in I}$ associated to patterns $\{\pi_i\}_{i\in I}$ of length k_i such that

- $R \subseteq \bigcup_{i\in I} W_i \subseteq S$
- for all $i \in I$, $Post_{\pi_i}(W_i, k_i\tau) \subseteq R$
- for all $i \in I$, $Tube_{\pi_i}(W_i) \subseteq S$

The sets W_i are obtained by repeated bisection or tiling. A first covering of R is generated (potentially a single set), and for each set of the covering, all possible control sequences of increasing lengths are tested up to a given maximum length. If a control sequence is found verifying the above properties for each set of the covering, then the algorithm has succeeded, otherwise, the covering sets that have not been associated to a control sequence are divided into sub-sets, and the procedure is repeated for the sub-sets, until all sets and sub-sets of the covering are associated to a control sequence verifying the above properties. The algorithm fails when a chosen maximum division depth is reached. The division heuristics is not a requirement, taking a uniform and sufficiently fine initial covering of R would be enough for computing a controller, the division heuristics simply ensures a more time-efficient computation.

Note that the sets can be of various shapes (boxes, zonotopes, ellipsoids). In this paper, the covering sets $(W_i)_{i\in I}$ are balls of $\mathbb{R}^n$ while the covered sets R and S are either boxes or balls of $\mathbb{R}^n$.

B. Computation of the Post and Tube operators using the Euler method on balls of $\mathbb{R}^n$

The computation of the Post and Tube operators can be performed with any reachability analysis tool. Since balls of $\mathbb{R}^n$ are required in the following, we use a ball-based approach introduced in [6] that relies on the Euler method. It is reliable and very fast, making the control synthesis computation feasible on the coupled ODE-PDE system, which would not be the case with more refined tools. It also requires very limited hypotheses on the dynamics [9], [7], contrary to approaches relying on *e.g.* incremental stability [12] or monotonicity [17].

The hypotheses are the following: for all $j \in U$ functions f_j are Lipschitz, which ensures the existence of solutions to system (6); and for all $j \in U$, functions f_j are *one-sided Lipschitz* (OSL). Given these hypotheses, three constants can be defined for each f_j: the Lipschitz constant, the OSL constant (which roughly speaking measures the contractivity of the system, it can be positive or negative), and a third constant measuring the maximum the norm of f_j on S. With these three constants, a guaranteed reachable ball and reachability tube are computed as follows.

Given an initial point $\tilde{z}_0 \in S$ and a mode $j \in U$, we define the following "Euler approximate solution" $\tilde{\phi}_j(t;\tilde{z}_0)$ for $t \in [0,\tau]$ by

$$\tilde{\phi}_j(t;\tilde{z}_0) = \tilde{z}_0 + t f_j(\tilde{z}_0). \tag{7}$$

Any trajectory $z(t;0,z_0,j)$ of system (6) for mode $j \in U$ and time $t \in [0,\tau]$ that starts at $z_0 \in B(\tilde{z}_0,\rho)$ at $t=0$ remains in a circular tube around $\tilde{\phi}_j(t;\tilde{z}_0)$ whose radius is given by analytical formulas that depend on ρ, t, and the three constants mentioned above. In other words, we can compute a set-based reachability tube for an entire starting ball using a simple Euler scheme and some analytical formulas. The result given in [6] is even stronger than that since the inclusion of $Tube_{\pi_i}(W_i)$ in S for the tiling based control synthesis method can be verified using tests at discrete instants, making the algorithm extremely time-efficient. The reachability tube computation is formalized in the following theorem.

Theorem 1: Given a sampled switched system satisfying the Lipschitz and OSL hypotheses, consider a point $\tilde{z}_0$ and a positive real ρ. We have, for all $z_0 \in B(\tilde{z}_0,\delta)$, $t \in [0,\tau]$ and $j \in U$:

$$z(t;0,z_0,j) \in B(\tilde{\phi}_j(t;\tilde{z}_0), \delta_j(\rho,t)),$$

where function $\delta_j(\rho,t)$ is given in Appendix A.
The proof is given in [6].

IV. Guaranteed control for the ODE-PDE system

In order to use a formal control synthesis method on system (3), we need to transform it into a finite dimensional one, and ensure that the dimension reduction error is handled. In Section IV-A, we explain in details the decomposition of the transient PDE state. In Section IV-B, we detail the dimension reduction of the PDE state. In Section IV-C, the control synthesis procedure is applied on the decomposed and reduced system. In Section IV-D, we detail the *ad-hoc* certified reduced basis that ensures that the reduction error does not grow within time. We finally apply the approach on an illustrative case study in Section IV-E. The main idea in our approach is the following:

- Observe that the finite dimensional state $\boldsymbol{\xi}$ can be formally controlled, but u cannot in its current form.
- Observe that, using the principle of superposition, the infinite dimensional state u can be decomposed as the sum of three terms

$$u = u^\infty + u_q + \psi \tag{8}$$

  where
  - $u^\infty \in H^1$ is the objective state,
  - $u_q \in L^1_{\text{loc}}([0,\infty[;H^1)$ is the steady state (*i.e.* the long term behaviour of the infinite dimensional state),

– $\psi \in L^1_{\text{loc}}([0,\infty[;H^1)$ is the transient state (*i.e.* the short term behaviour of the infinite dimensional state).

- Rewrite the stability constraints (4) for u according to these terms so they can be handled separately.
- Use a reduced order approximation $\tilde{\psi}$ of ψ, and compute it using a finite dimensional ODE on a vector $\tilde{\boldsymbol{\beta}} \in \mathbb{R}^K$.
- Observe that the terms u^∞ and $u_q(\cdot,t)$ introduce a tighter stability objective that can be attained with a reasonable objective u^∞ (*i.e.* when it is consistent with the source term f).
- Perform the tiling based control synthesis on the reduced state $\boldsymbol{z} = (\boldsymbol{\xi}, \tilde{\boldsymbol{\beta}})$, while simultaneously ensuring that u^∞, u_q, and the additional error terms are taken into account in the stability constraints (by the control law or using an efficient reduction technique).

Remark 1: The stability properties that can be ensured with standard control synthesis methods for infinite dimensional systems are multiple. One could seek to ensure the exponential stabilization of the (infinite dimensional) system state in L^∞ norm, H^∞ norm, and so on... In our case, we consider the L^2 norm which allows us to use dimension reduction methods that transform the infinite dimensional system into a finite dimensional one that can be controlled using balls of $\mathbb{R}^n$. Approaches based on the L^2 norm are standard techniques in the field of structural mechanics. The Euclidian norm of the (finite dimensional) reduced state is directly related to the L^2 distance of the infinite dimensional state. Therefore, the sets (balls) defined on the reduced space directly correspond to sets in the unreduced space of the PDE state.

A. Decomposition of the infinite state

Using the principle of superposition, the solution u of problem 1 can be written as the sum of three functions

$$u = u^\infty + u_q + \psi \tag{9}$$

where $u^\infty \in H^1$ is the objective, $u_q \in L^1_{\text{loc}}([0,\infty[;H^1)$ is the quasi-steady state and $\psi \in L^1_{\text{loc}}([0,\infty[;H^1)$ is the transient state. The steady state u_q is the solution of the following quasi-static problem defined for all $(x,t) \in \Omega \times [0,+\infty[$ by

$$\begin{cases} -\nabla \cdot (\kappa \nabla u_q)(x,t) = f(x,t) + \nabla \cdot (\kappa \nabla u^\infty)(x), \\ u_q(0,t) = \xi_1(t) - \xi_1^\infty, \\ u_q(L,t) = \xi_2(t) - \xi_2^\infty, \end{cases}$$

where ξ_1^∞ and ξ_2^∞ are the boundary conditions of u^∞. The transient state ψ is the solution of the heat problem with homogeneous Dirichlet boundary conditions defined for all $(x,t) \in \Omega \times [0,+\infty[$ by

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x,t) - \nabla \cdot (\kappa \nabla \psi)(x,t) = g(x, \boldsymbol{\xi}(t)) \\ \psi(0,t) = \psi(L,t) = 0, \quad t > 0, \\ \psi(x,0) = \psi^0(x), \end{cases} \tag{10}$$

with

$$g(x,\boldsymbol{\xi}(t)) = -\frac{\partial u_q}{\partial t}(x,t), \ \psi^0(x) = u^0(x) - u^\infty(x) - u_q(x,0).$$

The transient state $\psi(.,t)$ is thus in H^1_0 for any $t \geq 0$. The space weak formulation of the problem (10) is to find $\psi \in L^1_{\text{loc}}([0,\infty[;H^1_0)$, $\psi(.,0) = \psi^0$, solution of

$$\forall v \in H^1_0, \quad \langle \frac{\partial \psi}{\partial t}, v \rangle_{L^2} + \langle \kappa \nabla \psi, \nabla v \rangle_{L^2} = \langle g(\cdot, \boldsymbol{\xi}(\cdot)), v \rangle_{L^2} \tag{11}$$

Proposition 1 allows us to rewrite the infinite dimension state into a finite one with additional error terms.

Proposition 1: Consider Problem 1 and decomposition (9). For all $t > 0$, if

$$\frac{1}{\kappa_m} \| f + \nabla \cdot (\kappa \nabla u^\infty) \|_{L^2} + L \| \boldsymbol{\xi}(t) - \boldsymbol{\xi}^\infty \|_\infty + \| \psi(\cdot,t) \|_{L^2} \leq \rho, \tag{12}$$

then

$$\| u(\cdot,t) - u^\infty \|_{L^2} \leq \rho \quad \text{for all } t > 0. \tag{13}$$

The proof is given in Appendix B.

B. Reduction of the transient state ψ

In order to apply the control method detailed in section III, the transient state ψ must be approximated by a function $\tilde{\psi}$ that belongs to a finite dimensional vector space. To achieve this, we use a reduced-order model of ψ defined for all $(x,t) \in \Omega \times [0,+\infty[$ by

$$\tilde{\psi}(x,t) = \langle \tilde{\boldsymbol{\beta}}(t), \boldsymbol{\varphi}(x) \rangle_2 \tag{14}$$

where $\tilde{\boldsymbol{\beta}}(t) = (\tilde{\beta}_1(t), \ldots, \tilde{\beta}_K(t))^T$, $\boldsymbol{\varphi}(x) = (\varphi_1(x), \ldots, \varphi_K(x))^T$ with $\{\varphi^1, \ldots, \varphi^K\}$ being an L^2–orthogonal reduced basis spanning W^K, a linear vector space of dimension K. With the reduced approximation (14) we have for all $t > 0$,

$$\| \tilde{\psi}(.,t) \|_{L^2} = \| \tilde{\boldsymbol{\beta}}(t) \|_2.$$

In addition, $\tilde{\boldsymbol{\beta}}$ is the solution of an ordinary differential equation (see section IV-C) that can be controlled formally using the method given in section III. We thus have a reduced-order version of Proposition 1:

Proposition 2: Consider Problem 1, decomposition (9) and approximation (14). Suppose that there exists $\mu > 0$ such that for all $t \in [0,\tau]$,

$$\| \psi(\cdot,t) - \tilde{\psi}(\cdot,t) \|_{L^2} \leq \mu \, \| \psi^0 - \tilde{\psi}^0 \|_{L^2}. \tag{15}$$

If for all $t > 0$,

$$\frac{1}{\kappa_m} \| f + \nabla \cdot (\kappa \nabla u^\infty) \|_{L^2} + L \| \boldsymbol{\xi}(t) - \boldsymbol{\xi}^\infty \|_\infty + \| \tilde{\boldsymbol{\beta}}(t) \|_2 + \mu \, \| \psi^0 - \tilde{\psi}^0 \|_{L^2} \leq \rho, \tag{16}$$

then stability condition (13) holds.

The proof is given in Appendix C. Hypothesis (15) only supposes that the reduced basis approximation error does not explode within a time step. An efficient reduced basis approximation can furthermore ensure that for all $t \in [0,\tau]$, $\|\psi(\cdot,t)-\tilde{\psi}(\cdot,t)\|_{L^2} \leq \|\psi^0-\tilde{\psi}^0\|_{L^2}$ (*i.e.* $\mu=1$), which means that the approximation error does not grow within time. Such a reduced basis is constructed in Section IV-D. Inequality (16) means that all the terms in the left-hand side have to be "small enough". In particular, this means that u^∞ should be compatible with the source term in the sense that

$$-\nabla\cdot(\kappa\nabla u^\infty) \approx f \quad \text{ in } \Omega.$$

Moreover, the vector state $\boldsymbol{\xi}(t)$ should stay close to $\boldsymbol{\xi}^\infty$, and the norm of vector $\tilde{\boldsymbol{\beta}}(t)$ has to stay rather small. The terms $L\|\boldsymbol{\xi}(t)-\boldsymbol{\xi}^\infty\|_\infty$ and $\|\tilde{\boldsymbol{\beta}}(t)\|_2$ are actually the ones we control with our symbolic approach. Note that $L\|\boldsymbol{\xi}(t)-\boldsymbol{\xi}^\infty\|_\infty$ justifies that we stabilize $\boldsymbol{\xi}$ in a box. Finally, we should also have $\mu\|\psi^0-\tilde{\psi}^0\|_{L^2}$ small enough for any initial condition, meaning that the reduced basis approximation is able to correctly reproduce any initial condition. In a nutshell, we have to synthesize a controller such that the reduced state $\boldsymbol{z}=(\boldsymbol{\xi},\tilde{\boldsymbol{\beta}})$ always stays in $S=B_\infty(\xi^\infty,\delta_\xi)\times B_{L^2}(0,\rho_\beta)$ using symbolic methods (δ_ξ and ρ_β are chosen in the next subsection), and the other terms are fulfilled as long as the objective state is compatible with the source term, the reduced basis represents accurately the initial conditions, and the reduction error does not grow uncontrollably.

C. Strategy for control stability

At a switch time (considered equal to zero for the sake of simplicity), consider the approximate heat solution

$$\tilde{u}^0 = u^\infty + u_q(\cdot;\boldsymbol{\xi}^0)+\tilde{\psi}^0$$

and the exact solution written as

$$u^0 = u^\infty + u_q(\cdot;\boldsymbol{\xi}^0)+\psi^0.$$

Considering Problem 1, we assume the following initial properties. Let $\delta_\xi,\rho_\beta,\delta>0$ be some constants such that

$$L\|\boldsymbol{\xi}^0-\boldsymbol{\xi}^\infty\|_\infty \leq \delta_\xi, \tag{17}$$

$$\|\tilde{\boldsymbol{\beta}}^0\|_2 \leq \rho_\beta, \tag{18}$$

$$\mu\|\psi^0-\tilde{\psi}^0\|_{L^2} \leq \delta. \tag{19}$$

Suppose that

$$c_1+\delta_\xi+\rho_\beta+\delta \leq \rho \tag{20}$$

where $c_1=\frac{1}{\kappa_m}\|f+\nabla\cdot(\kappa(.)\nabla u^\infty)\|_{L^2}$. We look for controls that preserve equations (17) and (18) (and solve Problem 1). In other words, we look for control modes such that, for all time $t\in[0,\tau]$ (before the next switch), we have

$$\|\boldsymbol{\xi}(t)-\boldsymbol{\xi}^\infty\|_\infty \leq \frac{\delta_\xi}{L} \tag{21}$$

$$\text{and} \qquad \|\tilde{\boldsymbol{\beta}}(t)\|_2 \leq \rho_\beta. \tag{22}$$

Then by construction we will automatically fulfill the stability requirement (13) on the heat solution for a given control mode $i\in U$, *i.e.* for all $t\in[0,\tau]$,

$$\|u(\cdot,t)-u^\infty\|_{L^2} \leq \rho. \tag{23}$$

Equations (17) and (18) can also be ensured for control sequences $\pi=(\sigma_1,\sigma_2,\ldots,\sigma_k)$, *i.e.* for all $t\in[0,k\tau]$.

Both the state $\boldsymbol{\xi}$ and the reduced state $\tilde{\boldsymbol{\beta}}$ are solutions of ODEs (respectively of dimension 2 and K). Indeed, the reduced state $\tilde{\psi}$ is built as a solution of (11) with test functions spanning W^k, which reformulates as: for all $w\in W^K$,

$$\begin{cases}\langle\dfrac{\partial\widetilde{\psi}}{\partial t},w\rangle_{L^2}+\langle\kappa\nabla\widetilde{\psi},\nabla w\rangle_{L^2}=\langle g(\cdot;\boldsymbol{\xi}(t)),w\rangle_{L^2}\\ \tilde{\psi}(\cdot,0)=\tilde{\psi}^0.\end{cases} \tag{24}$$

The basis functions $\{\varphi^1,\ldots,\varphi^K\}$ being orthonormal in L^2, equation (24) is in fact the following system of ODEs

$$\begin{cases}\dfrac{d\tilde{\beta}_i}{dt}+\tilde{\beta}_i\langle\kappa\nabla\varphi_i,\nabla\varphi_j\rangle_{L^2}=\langle g(.;\boldsymbol{\xi}(t)),\varphi_j\rangle_{L^2},\\ 1\leq j\leq K.\end{cases} \tag{25}$$

To solve Problem 1, it is thus sufficient to synthesize a controller such that $\boldsymbol{z}=(\boldsymbol{\xi},\tilde{\boldsymbol{\beta}})$ always stays in $S=B_\infty(\xi^\infty,\frac{\delta_\xi}{L})\times B_2(0,\rho_\beta)$ using the algorithm of Section III (provided that hypothesis (15) holds). This algorithm is particularly adapted to this purpose since for all $t>0$, $\|\tilde{\psi}(.,t)\|_{L^2}=\|\tilde{\boldsymbol{\beta}}(t)\|_2$. The consequence is that ensuring that $\tilde{\boldsymbol{\beta}}$ remains in $B_2(0,\rho_\beta)$ also ensures that $\tilde{\psi}$ remains in $B_{L^2}(0,\rho_\beta)$. The recurrence set R can be chosen as any smaller set $R\subseteq S$ such that stability in S is ensured. From (21), it is appropriate to choose the safety set S_ξ in Problem 1 as $B_\infty(\xi^\infty,\frac{\delta_\xi}{L})$, *i.e.* a box centered around $\boldsymbol{\xi}^\infty$.

D. Certified reduced basis for control

To complete our method, we construct of an efficient reduced basis that allows to verify (16) (and more precisely, ensure (15) with $\mu=1$). Our objective is the following. Considering the space of all possible sequences of switched controls of lengths less than M, we have to derive a reduced-order model which guarantees a prescribed accuracy for any switched control sequence. For that purpose, we build a reduced-order model using a posteriori error estimates within an iterative greedy approach. Therefore, we consider a low-dimensional vector space $W\subset H_0^1$ and use the Galerkin method to build a reduced-order approximation $\tilde{\psi}$ of ψ that is a solution of the finite dimensional weak problem, for all $w\in W$

$$\begin{cases}\langle\dfrac{\partial\widetilde{\psi}}{\partial t},w\rangle_{L^2}+\langle\kappa\nabla\widetilde{\psi},\nabla w\rangle_{L^2}=\langle g(\cdot;\boldsymbol{\xi}(t)),w\rangle_{L^2},\\ \tilde{\psi}(.,0)=\tilde{\psi}^0.\end{cases}$$

1) A posteriori error estimation: From (11), we can directly derive a weak problem for the error function $e=\psi-\widetilde{\psi}$, for all $v\in H_0^1$,

$$\langle\frac{\partial e}{\partial t},v\rangle_{L^2}+\langle\kappa\nabla e,\nabla v\rangle_{L^2}=$$

$$\langle g(\cdot;\boldsymbol{\xi}(t)),v\rangle_{L^2}-\langle\frac{\partial\widetilde{\psi}}{\partial t},v\rangle_{L^2}-\langle\kappa\nabla\widetilde{\psi},\nabla v\rangle_{L^2} \tag{26}$$

$$e(\cdot,0)=e^0=\psi^0-\widetilde{\psi}^0. \tag{27}$$

The right hand side defines a residual linear form r_ξ that depends on $\boldsymbol{\xi}(t)$, for all $v \in H_0^1$,

$$r_\xi(v) = \langle g(\cdot;\boldsymbol{\xi}(t)), v\rangle_{L^2} - \langle \frac{\partial \widetilde{\psi}}{\partial t}, v\rangle_{L^2} - \langle \kappa \nabla \widetilde{\psi}, \nabla v\rangle_{L^2}.$$

By construction of the approximation $\widetilde{\psi}$ (equation (24)) we have for all $w \in W$,

$$r_\xi(w) = 0.$$

One can define a norm for r_ξ in the dual space H^{-1} of H_0^1 :

$$\|r_\xi\|_{H^{-1}} = \sup_{\|v\|_{H_0^1} \leq 1} |r_\xi(v)|.$$

By defining the error e and the residual r_ξ we can derive the following proposition.

Proposition 3: Consider equation (26) and (27). If

$$\frac{\tilde{\eta}\, C_\Omega^2}{\kappa_m} \leq \|e_0\| \tag{28}$$

where C_Ω is a Poincaré constant and

$$\tilde{\eta} = \sup_{\boldsymbol{\xi} \in \mathbb{R}^2} \sup_{t \geq 0} \|r_\xi\|_{H^{-1}}(t),$$

then for all $t > 0$

$$\|e(t)\|_{L^2} \leq \|e(0)\|_{L^2}.$$

The proof is given in Appendix D.

Remark 2: Because the approximate problem is built from a Galerkin projection method, it is expected that the constant $\tilde{\eta}$ becomes small for a "good" finite discrete space W. So for an accuracy level $\|e_0\|_{L^2} \leq \delta$ on the initial state, the goal is to find a discrete reduced-order space W such that the inequality $\tilde{\eta} \leq \frac{\kappa_m \delta}{C_\Omega^2}$ holds. The constant $\tilde{\eta}$ defined in (41) is a uniform upper bound of the residual quantity, meaning that $\tilde{\eta}$ should be rather small for any switched control sequence $\sigma(.)$ for practical use. This remark leads us to the following greedy algorithm for the construction of the reduced order basis.

2) Greedy algorithm and reduced bases: The greedy algorithm allows to compute a reduced basis that spans the discrete space $\tilde{W}$ in an iterative and greedy manner.

- First iterate $k = 1$. Define $\delta > 0$ and a residual threshold

  $$r_M = \frac{\kappa_m \delta}{C_\Omega^2}.$$

  Let us assume that $\psi \in H_0^1$ and $\psi^0 \neq 0$. Let us consider first

  $$\varphi^1 = \frac{\psi^0}{\|\psi^0\|}$$

  and $W^{(1)} = \mathrm{Span}(\varphi^1)$. Define a random sequence of control sequences $\sigma \in \Sigma^\tau$, *i.e.* control sequences of length less than K. As soon as

  $$\|r_\xi\|_{H^{-1}}(t) > r_M,$$

  solve the reduced-order model, for all $w \in W^{(1)}$

  $$\langle \frac{\partial \widetilde{\psi}^{(1)}}{\partial t}, w\rangle + \langle \kappa \nabla \widetilde{\psi}^{(1)}, \nabla w\rangle = \langle g(\cdot;\boldsymbol{\xi}(t)), w\rangle, \tag{29}$$

  $$\widetilde{\psi}^{(1)}(\cdot,0) = \tilde{\psi}^0. \tag{30}$$

- If there is a time $t^{(1)} > 0$ such that $\|r_\xi\|_{H^{-1}}(t^{(1)}) = r_M$, then compute

  $$v^{(2)} = \arg\max_{\|v\|=1} |r_{\xi(t^{(1)})}(v)|$$

  and define

  $$\varphi^2 = \frac{v^{(2)}}{\|v^{(2)}\|}, \quad W^{(2)} = \mathrm{Span}(\varphi^1, \varphi^2).$$

- The reduced-order model at iterate (k) is

  $$\langle \frac{\partial \widetilde{\psi}^{(k)}}{\partial t}, w\rangle + \langle \kappa \nabla \widetilde{\psi}^{(k)}, \nabla w\rangle = \langle g(\cdot;\boldsymbol{\xi}(t)), w\rangle, \tag{31}$$

  $$\widetilde{\psi}^{(k)}(\cdot,0) = \tilde{\psi}^0. \tag{32}$$

- Repeat until $\|r_\xi\|_{H^{-1}} < r_M$ for all time $t > 0$. Let us denote by K the final rank and $W^{(K)} = \mathrm{Span}(\varphi^1, \varphi^2, ..., \varphi^K)$ the associated discrete space.

For performance and complexity aspects, the rank K is expected to be not too large. For that, the initial accuracy radius δ should be chosen not to small.

E. Numerical experiment for the L^2 guaranteed control synthesis by stability of error balls

As a proof of concept, we apply the strategy described here on system (3) with 4 switched modes $A_\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{b}_\sigma \in \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ with a time step $\tau = 0.05$, κ chosen constant equal to 1. The reduced basis is built using the procedure of Section IV-D. This reduced basis allows to fulfill Proposition 2 with $\mu = 1$. The reduced basis is truncated at $K = 4$ modes. Associated to the ODE, we thus get a reduced system of dimension 6. Using control sequences of length 8, and a decomposition of the reduced state-space in $4^6 = 4096$ balls, we manage to synthesize a controller in approximately 20 minutes, with an objective state $(\xi^\infty, u^\infty) = (0_{\mathbb{R}^2}, 0_{L^2})$ and guaranteed L^2 error of $\rho = 0.5$. A simulation of the controller is given in Figure 1, where the initial condition is set as a random combination of the first ten eigenmodes and a lifting, such that (17-19) holds with $\delta_\xi = 0.2$, $\rho_\beta = 0.2$ and $\delta = 0.1$.

V. Conclusion

We have presented a formal method for stabilizing a coupled ODE-PDE system. Mathematical transformations allow to exhibit controllable terms, and a reduced order approximation of the PDE state allows to exhibit finite dimensional terms that can be formally controlled using a known tiling based control synthesis algorithm. Contrary to methods proposed in the past, we do not use an explicit discretization of the PDE before tackling the problem, and thus, we do not rely on FEM discretization error bounds. We propose an ad-hoc reduced basis that allows to prevent the model reduction error from growing in time, but any method preventing the reduction error from growing uncontrollably could work.

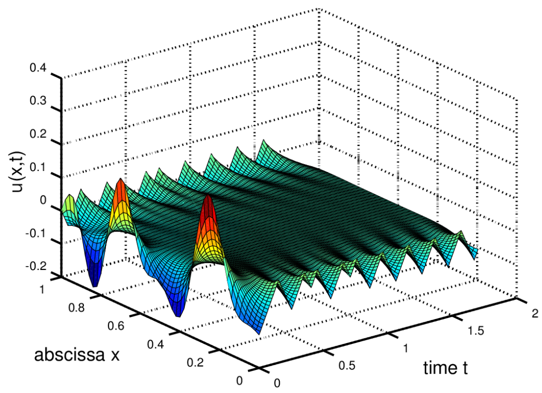

Fig. 1. Simulation of the controller.

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APPENDIX

A. Formulas of the Euler based reachability tubes

Consider a sampled switched system verifying the Lipschitz and OSL hypotheses. The three constants to compute (for each switched mode) are denoted by L_j, C_j, λ_j ($j \in U$). Their computation is realized with a constrained optimization algorithm. They are performed using the "sqp" function of Octave, applied on the following optimization problems:

- Constant L_j:

$$L_j = \max_{x,y \in S,\ x \neq y} \frac{\|f_j(y) - f_j(x)\|}{\|y - x\|}$$

- Constant C_j:

$$C_j = \max_{x \in S} L_j \|f_j(x)\|$$

- Constant λ_j:

$$\lambda_j = \max_{x,y \in T,\ x \neq y} \frac{\langle f_j(y) - f_j(x), y - x \rangle}{\|y - x\|^2}$$

Let ρ be a positive constant. For all $0 \leq t \leq \tau$ and all $j \in U$, function $\delta_j(\rho,t)$ is defined as follows:

- if $\lambda_j < 0$:

$$\delta_j(\rho,t) = \left(\rho^2 e^{\lambda_j t} + \frac{C_j^2}{\lambda_j^2} \left(t^2 + \frac{2t}{\lambda_j} + \frac{2}{\lambda_j^2} \left(1 - e^{\lambda_j t} \right) \right) \right)^{\frac{1}{2}}$$

- if $\lambda_j = 0$:

$$\delta_j(\rho,t) = \left(\rho^2 e^t + C_j^2(-t^2 - 2t + 2(e^t - 1)) \right)^{\frac{1}{2}}$$

- if $\lambda_j > 0$:

$$\delta_j(\rho,t) =$$
$$\left(\rho^2 e^{3\lambda_j t} + \frac{C_j^2}{3\lambda_j^2} \left(-t^2 - \frac{2t}{3\lambda_j} + \frac{2}{9\lambda_j^2} \left(e^{3\lambda_j t} - 1 \right) \right) \right)^{\frac{1}{2}}$$

Note that $\delta_j(\rho,t) = \rho$ for $t = 0$.

More details on this method are given in [6].

B. Proof of Proposition 1

Because of (9), the stability requirement for all $t > 0$, $\|u(\cdot,t) - u^\infty(.)\|_{L^2} \leq \rho$ in (4) can be equivalently expressed as for all $t > 0$, $\|u_q(\cdot,t) + \psi(.,t)\|_{L^2} \leq \rho$. The solution u_q itself can be decomposed (using the principle of superpositon) as

$$u_q(\cdot,t) = \bar{u} + w_q(\cdot,t),$$

where $\bar{u}$ is solution of the steady elliptic problem with homogeneous Dirichlet boundary conditions

$$-\nabla\cdot(\kappa(.)\nabla\bar{u}) = f + \nabla\cdot(\kappa\nabla u^\infty) \quad \text{in } \Omega, \quad (33)$$
$$\bar{u}(0) = \bar{u}(L) = 0, \quad (34)$$

and w_q is solution of the quasi-static problem at each time t:

$$-\nabla\cdot(\kappa(.)\nabla w_q) = 0 \text{ in } \Omega, \quad (35)$$
$$w_q(0,t) = \xi_1(t) - \xi_1^\infty, \quad \text{for all } t > 0, \quad (36)$$
$$w_q(L,t) = \xi_2(t) - \xi_2^\infty, \quad \text{for all } t > 0. \quad (37)$$

The solution $\bar{u}$ is continuous with respect to the source term in (33) [11], *i.e.* there exists $C > 0$ such that:

$$\|\bar{u}\|_{H_0^1} \leq C\,\|f + \nabla\cdot(\kappa\nabla u^\infty)\|_{L^2}. \quad (38)$$

This inequality is a result of the Lax-Milgram theorem, and constant C is the inverse of the coercivity constant of the PDE problem, in this case $C = \frac{1}{\kappa_m}$. For the solution w_q of (35)-(37), because of the maximum principle [16], we have

$$\|w_q(.,t)\|_{L^\infty(\Omega)} = \max(|\xi_1(t) - \xi_1^\infty|, |\xi_2(t) - \xi_2^\infty|) \quad (39)$$
$$= \|\boldsymbol{\xi}(t) - \boldsymbol{\xi}^\infty\|_\infty. \quad (40)$$

Thus,

$$\|u_q(\cdot,t) + \psi(\cdot,t)\|_{L^2} \leq \|\bar{u}\|_{L^2} + \|w_q\|_{L^2} + \|\psi(\cdot,t)\|_{L^2}$$
$$\leq \|\bar{u}\|_{L^2} + L\|w_q\|_{L^\infty} + \|\psi(\cdot,t)\|_{L^2},$$

and finally

$$\|u_q(\cdot,t) + \psi(\cdot,t)\|_{L^2} \leq \frac{1}{\kappa_m}\|f + \nabla\cdot(\kappa\nabla u^\infty)\|_{L^2} + L\|\boldsymbol{\xi}(t) - \boldsymbol{\xi}^\infty\|_\infty + \|\psi(\cdot,t)\|_{L^2}$$

A sufficient condition to satisfy the stability constraint (13) is then to fulfill (12).

C. Proof of Proposition 2

By the triangular inequality we can write

$$\|\psi(\cdot,t)\|_{L^2} \leq \|\psi(\cdot,t) - \tilde{\psi}(\cdot,t)\|_{L^2} + \|\tilde{\psi}(\cdot,t)\|_{L^2}$$
$$\leq \|\psi(\cdot,t) - \tilde{\psi}(\cdot,t)\|_{L^2} + \|\tilde{\boldsymbol{\beta}}(t)\|_2.$$

Let us assume that we have the stability estimate for the reduced-order approximation: there exists a constant $\mu > 0$ such that

$$\|\psi(\cdot,t) - \tilde{\psi}(\cdot,t)\|_{L^2} \leq \mu\,\|\psi^0 - \tilde{\psi}^0\|_{L^2} \quad \forall t \in [0,\tau]$$

for any constant control mode $\sigma \in \{1,...,M\}$ (uniform stability with respect to the controls). This hypothesis can actually be verified with a proper construction of the reduced basis. Then, a more restrictive sufficient condition to fulfill the stability constraint (13) is to verify

$$\frac{1}{\kappa_m}\|f + \nabla\cdot(\kappa\nabla u^\infty)\|_{L^2} + L\|\boldsymbol{\xi}(t) - \boldsymbol{\xi}^\infty\|_\infty + \|\tilde{\boldsymbol{\beta}}(t)\|_2 + \mu\,\|\psi^0 - \tilde{\psi}^0\|_{L^2} \leq \rho.$$

This equation is interesting since it enlightens the different controllable and uncontrollable terms.

Let us denote by $P_K : H_0^1 \to W^K$ the continuous linear orthogonal projection operator over the low-order space W^K. Still by a triangular inequality, we have

$$\|\psi^0 - \tilde{\psi}^0\|_{L^2} \leq \|\psi^0 - P_K\psi^0\|_{L^2} + \|P_K\psi^0 - \tilde{\psi}^0\|_{L^2},$$

The projection $P_K\psi^0$ is given by

$$P_K\psi^0 = \sum_{k=1}^{K}\beta_k^0\,\varphi^k,$$

with $\beta_k^0 = (\psi^0,\varphi^k)_{L^2}$, $k = 1,...,K$. By denoting $\boldsymbol{\beta}^0 = (\beta_1^0,...,\beta_K^0)$, we then have

$$\|\psi^0 - \tilde{\psi}^0\|_{L^2} \leq \|\psi^0 - P_K\psi^0\|_{L^2} + \|\boldsymbol{\beta}^0 - \tilde{\boldsymbol{\beta}}^0\|_2,$$

D. Proof of Proposition 3

Considering the particular test function $v = e$ in (26)-(27), we have

$$\frac{1}{2}\frac{d}{dt}\|e\|_{L^2}^2 + \|\kappa\nabla e\|_{L^2}^2 = r_\xi(e).$$

From Poincaré's inequality

$$\|v\|_{L^2} \leq C_\Omega\|\nabla v\|_{L^2} \quad \forall v \in H_0^1$$

and the lower bound κ_m of κ, we have also

$$\frac{1}{2}\frac{d}{dt}\|e\|_{L^2}^2 \leq -\frac{\kappa_m}{C_\Omega^2}\|e\|_{L^2}^2 + \|r_\xi\|_{H^{-1}}(t)\,\|e\|_{L^2}.$$

Let us denote the constant

$$\tilde{\eta} = \sup_{\boldsymbol{\xi}\in\mathbb{R}^2}\ \sup_{t\geq 0}\ \|r_\xi\|_{H^{-1}}(t) \quad (41)$$

with $\sigma \in \Sigma^\tau$ such that $\xi(t) \in S_\xi$ for all $t \geq 0$ and ξ solution to

$$\dot{\boldsymbol{\xi}} = A_\sigma\xi + B\boldsymbol{w}_\sigma, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}^0.$$

So we have the estimation

$$\frac{1}{2}\frac{d}{dt}\|e\|_{L^2}^2 \leq -\frac{\kappa_m}{C_\Omega^2}\|e\|_{H_0^1}^2 + \tilde{\eta}\,\|e\|_{L^2}. \quad (42)$$

By using the Young inequality

$$\tilde{\eta}\,\|e(t)\|_{L^2} \leq \frac{\kappa_m}{2C_\Omega^2}\|e(t)\|_{L^2}^2 + \frac{C_\Omega^2}{2\kappa_m}\tilde{\eta}^2$$

and Gronwall's lemma to the resulting estimate, we get the error estimate in L^2-norm

$$\|e(t)\|_{L^2}^2 \leq \exp(-\frac{\kappa_m}{C_\Omega^2}t)\|e^0\|_{L^2}^2 + \frac{\tilde{\eta}^2C_\Omega^4}{\kappa_m^2}\left(1 - \exp(-\frac{\kappa_m}{C_\Omega^2}t)\right). \quad (43)$$

Hence, assuming

$$\frac{\tilde{\eta}\,C_\Omega^2}{\kappa_m} \leq \|e_0\|$$

leads to the result.